XVII. *On the Method of determining, from the real Probabilities of Life, the Values of contingent Reverfions in which Three Lives are involved in the Survivorfhip. By Mr.* William Morgan, *F. R. S.*

Read May 26, 1791.

HAVING been encouraged to the further purfuit of the doctrine of furvivorfhips by the very honourable manner in which my two former Papers on this fubject were received by the Royal Society, I think it my duty to fubmit the refult of my labours to their confideration. The folutions of fome of the following problems might have been derived from thofe which I have already communicated; but the direct inveftigation of each feparate problem being certainly more fatisfactory, and the rules obtained by this means in general more fimple, I have confidered no problem as connected with another, except the relation between them either immediately arifes from the folution, or is neceffary to prove the truth of it. Being anxious to render myfelf as concife as poffible, I have been minute only in the inveftigation of the firft problem, and have done little more than ftate the contingencies which will determine the furvivorfhip in the others. By the affiftance, however, of thefe, and the operations which are detailed in my former Papers, the theorems which I have given may be deduced without much difficulty.

In

In order to prevent unneceſſary repetitions, it may not be improper to begin with explaining the different ſymbols which are uſed in the following pages.

A, B, C, } denote the value of an annuity on the reſpective lives of A, B, or C.

D, denotes the value of S on the contingency of C's ſurviving A (by my 2d prob. Vol. LXXVIII).

E, denotes the ſame value on the contingency of B's ſurviving A, found by the ſame problem.

F, denotes the value of an annuity on a life one year younger than B.

G, denotes the value of the abſolute reverſion of S after the death of A.

H, denotes the value of an annuity on a life one year younger than A.

K, denotes the value of an annuity on a life one year younger than C.

L, denotes the value of an annuity on the longeſt of the three lives of A, B, and C.

M, denotes the value of S by the firſt problem on the contingency that A's life ſhall be the *firſt* that fails.

N, denotes the value of an annuity on a life one year older than A.

P, denotes the value of an annuity on a life one year older than B.

R, denotes the value of S on the contingency of B's dying after A (by my 3d prob. Vol. LXXVIII).

S, denotes the given ſum.

T, denotes the value of an annuity on a life one year older than C.

V, denotes

V, denotes the perpetuity.

W, denotes the value of S on the contingency of C's dying after A (by my 3d prob. Vol. LXXVIII).

α and a denote the number of perſons living in a table of obſervations at the ages of H and A.

s, t, u, w, &c. denote the number of perſons living at the end of the 1ſt, 2d, 3d, &c. years from the age of A.

β and b, denote the number of perſons living at the ages of F and B.

m, n, o, p, &c. denote the number of perſons living at the end of the 1ſt, 2d, 3d, &c. years from the age of B.

$\varkappa$ and c denote the number of perſons living at the ages of K and C.

d, e, f, g, &c. denote the number of perſons living at the end of the 1ſt, 2d, 3d, &c. years from the age of C.

a', a'', a''', &c. }
b', b'', b''', &c. } denote the decrements of life at the end of the 1ſt, 2d, 3d, &c. years from the
c', c'', c''', &c. } reſpective ages of A, B, and C.

r, denotes the value of £. 1 increaſed by its intereſt for a year.

The combinations of two or three of the ſeveral letters A, B, C, F, H, &c. denote the values of annuities on the joint continuance of two or three of thoſe reſpective lives.

PROBLEM I.

To determine the value of a given ſum, payable if A ſhould be the *firſt* that fails of the three lives A, B, and C.

SOLUTION.

In order to receive the given ſum in the firſt year, it is neceſſary that one or other of four events ſhould happen. 1ſt, That

That all the three lives ſhould fail, and that A ſhould die firſt. 2dly, That B ſhould die *after* A, and C live. 3dly, That C ſhould die *after* A, and B live. 4thly, That A only ſhould die, and B and C both live. Theſe ſeveral contingencies being expreſſed by $\frac{a' . \overline{b-m} . \overline{c-d}}{3abc}$, $\frac{a' . \overline{b-m} . d}{2abc}$, $\frac{a' . \overline{c-d} . m}{2abc}$, and $\frac{a' . dm}{abc}$, reſpectively, their ſum will be $= \frac{1}{abc} \times \overline{\frac{a'bc}{3} + \frac{a'md}{3} + \frac{a'mc}{6} + \frac{a'bd}{6}}$. In the ſecond and following years one or other of the ſame events muſt take place in order to receive the given ſum; that is, they muſt either all three die, A dying firſt; or only A and B muſt die, A dying firſt; or only A and C muſt die, A dying firſt; or only A muſt die, and B and C both live. The different fractions expreſſing thoſe four contingencies for the ſecond year being $\frac{a'' . \overline{m-n} . \overline{d-e}}{3abc}$, $\frac{a'' . \overline{m-n} . e}{2abc}$, $\frac{a'' . \overline{d-e} . n}{2abc}$, and $\frac{a'' . ne}{abc}$; for the third year $\frac{a''' . \overline{n-o} . \overline{e-f}}{3abc}$, $\frac{a''' . \overline{n-o} . f}{2abc}$, $\frac{a''' . \overline{e-f} . o}{2abc}$, and $\frac{a''' . of}{abc}$, and ſo on, the whole value of the given ſum will be $=$

$$\frac{S . a'}{abcr} \times \overline{\frac{bc}{3} + \frac{md}{3} + \frac{mc}{6} + \frac{bd}{6}} + \frac{S . a''}{abcr^2} \times \overline{\frac{md}{3} + \frac{ne}{3} + \frac{nd}{6} + \frac{me}{6}} + \frac{S . a'''}{abcr^3} \times \overline{\frac{ne}{3} + \frac{of}{3} + \frac{oe}{6} + \frac{nf}{6}} + \&c.$$

which are $= \frac{S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a''' . en}{r^3} + \&c.} +$

$$\frac{S}{6abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a''' . oe}{r^3} + \&c.} + \frac{S}{6abc} \times \overline{\frac{a'db}{r} + \frac{a''em}{r^2} + \frac{a''' . fn}{r^3} + \&c.}$$

$$+ \frac{S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a'' . en}{r^2} + \frac{a'''fo}{r^3} + \&c.}$$

From the demonſtration of the problem in my laſt Paper * it appears, that the firſt of theſe ſeries is $= \frac{S}{3} \times \frac{\beta\kappa . \overline{FK - AFK}}{bc} - \frac{S}{3r} \times \overline{BC - ABC}$; that the ſecond ſeries is $= \frac{S}{6} \times \frac{\kappa . \overline{BK - ABK}}{c} - \frac{S}{6r} \times \frac{m . \overline{PC - APC}}{b}$; that the

* See Phil. Tranſ. Vol. LXXIX.

 third

third series is $= \frac{S}{6} \times \frac{\beta \,.\, \overline{FC - AFC}}{b} - \frac{S}{6r} \times \frac{d \,.\, \overline{BT - ABT}}{c}$; and that the fourth series is $= \frac{S}{3} \times \overline{BC - ABC} - \frac{S}{3r} \times \frac{md \,.\, \overline{PT - APT}}{bc}$. These several expressions being added together will be found $=$ S into $\frac{x}{3c} \times \overline{\frac{\beta \,.\, \overline{FK - AFK}}{b} + \frac{BK - ABK}{2}} + \frac{\beta}{6b} \times \overline{FC - AFC} + \frac{-1}{3r} \times \overline{BC - ABC} - \frac{m \,.\, \overline{PC - APC}}{6br} - \frac{d}{3cr} \times \overline{\frac{BT - ABT}{2} + \frac{m \,.\, \overline{PT - APT}}{b}}$.

This theorem gives the exact value when either B or C are the oldest of the three lives; but when A is the oldest, it will be necessary to exchange the symbols a', a'', a''', &c. for $\overline{a - s}$, $\overline{s - t}$, $\overline{t - u}$, &c. and the symbols m, n, o, &c. for $\overline{c - c'}$, $\overline{c - c' + c''}$, $\overline{c - c' + c'' + c'''}$, &c. In this case the value of the given sum for the first year will be found $= \frac{S}{abcr}$ into $\frac{abc}{2} - \frac{bsc}{2} + \frac{amc}{2} - \frac{msc}{2} - \frac{abc'}{6} + \frac{bsc'}{6} - \frac{amc'}{3} + \frac{msc'}{3}$; for the second year $= \frac{S}{abcr^2}$ into $\frac{msc}{2} - \frac{mtc}{2} + \frac{nsc}{2} - \frac{ntc}{2} - \frac{msc''}{6} + \frac{mtc''}{6} - \frac{nsc''}{3} + \frac{ntc''}{3} - \frac{msc'}{2} + \frac{mtc'}{2} - \frac{nsc'}{2} + \frac{ntc'}{2}$; for the third year $= \frac{S}{abcr^3}$ into $\frac{ntc}{2} - \frac{nuc}{2} + \frac{otc}{2} + \frac{ouc}{2} - \frac{ntc'''}{6} + \frac{nuc'''}{6} - \frac{otc'''}{3} + \frac{ouc'''}{3} - \frac{nt \,.\, \overline{c' + c''}}{2} + \frac{nu \,.\, \overline{c' + c''}}{2} - \frac{ot \,.\, \overline{c' + c''}}{2} + \frac{ou \,.\, \overline{c' + c''}}{2}$; and so on for the remaining years of A's life. Hence the whole value of the given sum will be $= \frac{S}{2ab} \times \overline{\frac{ab}{r} + \frac{ms}{r^2} + \frac{nt}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{bs}{r} + \frac{mt}{r^2} + \frac{nu}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{am}{r} + \frac{ns}{r^2} + \frac{ot}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{ms}{r} + \frac{nt}{r^2} + \frac{ou}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{msc'}{r} + \frac{nt \,.\, \overline{c' + c''}}{r^2} + \&c.} + \frac{S}{6abc} \times \overline{\frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c.} + \frac{S}{2abcr} \times$

mtc'

$\overline{\frac{mtc'}{r} + \frac{nu \,.\, \overline{c'+c''}}{r^2} + \&c.} - \frac{S}{3abc} \times \overline{\frac{amc'}{r} + \frac{ns \,.\, c''}{r^2} + \frac{ot \,.\, c'''}{r^3} + \&c.} - \frac{S}{2abcr} \times$ $\overline{\frac{nsc'}{r} + \frac{ot \,.\, \overline{c'+c''}}{r^2} + \&c.} + \frac{S}{3abc} \times \overline{\frac{msc'}{r} + \frac{nt \,.\, c''}{r^2} + \frac{ouc'''}{r^3} + \&c.} \times \frac{S}{2abcr} \times$ $\overline{\frac{nt \,.\, c'}{r} + \frac{ou \,.\, \overline{c'+c''}}{r^2} + \&c.}$ . . . The first four series are respectively $=$ $\frac{\alpha\beta \,.\, HF}{2ab}$ $\left(\text{or } \frac{1+AB}{2r}\right) - \frac{\beta \,.\, AF}{2b} + \frac{\alpha \,.\, HB}{2a} - \frac{AB}{2}$ . . . The first term of the fifth series, or $\frac{abc'}{r}$, is $= \frac{\alpha\beta}{abr} \times \overline{\frac{ab}{\alpha\beta} - \frac{ab \,.\, \overline{c-c'}}{\alpha\beta c}}$, the second term or $\frac{msc''}{r^2}$, is $= \frac{\alpha\beta}{abr^2} \times \overline{\frac{ms}{\alpha\beta} - \frac{ms \,.\, \overline{c-\overline{c'+c''}}}{\alpha\beta c}} - \frac{msc'}{r^2}$ . . . . the third term, or $\frac{ntc'''}{r^3}$, is $= \frac{\alpha\beta}{abr^3} \times \overline{\frac{nt}{\alpha\beta} - \frac{nt \,.\, \overline{c-\overline{c'+c''+c'''}}}{\alpha\beta c}} - \frac{nt' \,.\, \overline{c'+c''}}{r^3}$. Therefore the sum of the fifth and sixth series is $= -\frac{\alpha\beta \,.\, \overline{HF-HFC}}{6ab}\left(-\frac{msc'}{3abcr^2} - \frac{nt \,.\, \overline{c'+c''}}{3abcr^3} - \&c. =\right) - \frac{1}{3r} \times \overline{AB - ABC}$. In the same manner the seventh series may be found $= \frac{\beta}{6b} \times \overline{AF - AFC} - \frac{mtc'}{6abcr^2} - \frac{nu \,.\, \overline{c+c''}}{6abcr^3} - \&c.$; and consequently the sum of the seventh and eighth series is $= \frac{\beta \,.\, \overline{AF-AFC}}{6b} + \frac{s \,.\, \overline{BN-BNC}}{3ar}$. Again, the ninth and tenth series may be found $= -\frac{\alpha \,.\, \overline{HB-HBC}}{3a}\left(-\frac{nsc'}{6abcr^2} - \frac{ot \,.\, \overline{c'+c''}}{6abcr^3}\right.$ $\left.- \&c. =\right) - \frac{m \,.\, \overline{AP-APC}}{6br}$; and the eleventh and twelfth or two last series, $= \frac{AB-ABC}{3}\left(+\frac{ntc'}{6abcr^2} + \frac{uo \,.\, \overline{c'+c''}}{6abcr^3} + \&c. =\right) + \frac{ms \,.\, \overline{PN-PNC}}{6abr}$. If all these expressions be added together, we shall have the value of the given sum $=$ S into $\overline{\frac{\beta}{3b} \times \frac{\alpha \,.\, HF + \frac{1}{2}HFC}{a} - AF + \frac{1}{2}AFC}$

$$+\frac{1}{6}\times\overline{\frac{\alpha.\overline{HB+2HBC}}{a}-\overline{AB+2ABC}}+\frac{1}{3r}-\overline{\frac{s.\overline{BN-BNC}}{a}-\overline{AB-ABC}}$$

$$+\frac{m}{6br}\times\overline{\frac{s.\overline{PN-PNC}}{a}-\overline{AP-APC}}.$$

It may be obferved, that the firft four feries are alfo $=$ E, and therefore if this value be fubftituted inftead of $\frac{\alpha\beta.HF}{2ab}$, &c. the general rule will become $= E+S$ into $-\frac{\alpha}{3a}\times\overline{HB-HBC}+\overline{\frac{\beta}{2b}\times\overline{HF-HFC}+\frac{\overline{r-1}.\overline{AB-ABC}}{3r}+\frac{\beta.\overline{AF-AFC}}{6b}-\frac{m.\overline{AP-APC}}{6br}}+\overline{\frac{s}{3ar}\times\overline{BN-BNC}+\frac{2b}{m}\times\overline{PN-PNC}}$ ...... Suppofing the three lives to be equal, the firft of thefe rules will become $=\frac{1+CC}{2r}-\frac{\kappa\kappa.\overline{KK-CKK}}{6.cc}-\frac{1}{3r}\times\overline{CC-CCC}-\frac{\kappa.\overline{CK-CCK}}{6c}+\frac{d.\overline{CT-CCT}}{6cr}-\frac{\overline{CC+2CCC}}{6}+\frac{dd.\overline{TT-CTT}}{6.cc.r}$, and the fecond $=\frac{\overline{r-1}.\overline{V-CC}}{2r}+\frac{\overline{r-1}.\overline{CC-CCC}}{3r}-\frac{\kappa.\overline{CK-CCK}}{6c}-\frac{\kappa\kappa.\overline{KK-CCK}}{6cc}+\frac{d.\overline{CT-CCT}}{6cr}+\frac{dd.\overline{TT-CTT}}{6ccr}$: and alfo the rule denoting the value (when either B or C are the eldeft of the three lives) will become $=\frac{\overline{r-1}.\overline{CC-CCC}}{3r}+\frac{\kappa}{3c}\times\overline{CK-CCK}+\frac{\kappa\kappa}{3.cc}\times\overline{KK-CKK}-\frac{d}{c}\times\frac{\overline{CT-CCT}}{3r}-\frac{dd.}{3ccr}\times\overline{TT-CTT}$. Let this laft expreffion be called Q and compared with the firft of the preceding expreffions. In this cafe we fhall have $\frac{1+CC}{2r}-\frac{\overline{CC+2CCC}}{6}-\frac{\overline{CC-CCC}}{3r}-\frac{Q}{2}+\frac{\overline{r-1}.\overline{CC-CCC}}{6r}=Q$, from which Q may be eafily found $=\frac{S}{3}\times\frac{\overline{r-1}.\overline{V-CCC}}{r}$. Again, let the fame expreffion, denoted by by Q, be compared with the second of the preceding expressions, and we shall then have $\frac{\overline{r-1}\,.\,\overline{V-CC}}{2r} - \frac{Q}{2} + \frac{\overline{r-1}\,.\,\overline{CC-CCC}}{2r}$ $=Q$, and consequently $Q = S \times \frac{\overline{r-1}\,.\,\overline{V-CCC}}{3r}$ as before * .... Now it is well known, that when the lives are all equal, the value of the reversion must be *one third* the difference between the perpetuity and the three joint lives, and therefore a demonstration arises of the truth of the whole solution. As a still further proof of this, the foregoing theorem may be immediately deduced from the series themselves: thus, the value of the given sum for the first year will in this case be $= \frac{S\,.\,\overline{c^3-d^3}}{3c^3\,.\,r}$; for the second year it will be $= \frac{S\,.\,\overline{d^3-e^3}}{3c^3r^2}$; for the third year it will be $= \frac{S\,.\,\overline{e^3-f^3}}{3c^3r^3}$. Hence the value of the whole reversion will be $=$

$$\frac{S}{3} \times \overline{\frac{1}{r} - \frac{d^3}{c^3r} - \frac{e^3}{c^3r^2} - \&c.} + \frac{S}{3} \times \overline{\frac{d^3}{c^3r^2} + \frac{e^3}{c^3r^3} + \frac{f^3}{c^3r^4} + \&c.} = \frac{S}{3} \times \overline{\frac{1}{r} - CCC + \frac{CCC}{r}} = \frac{S}{3} \times \frac{\overline{r-1}\,.\,\overline{V-CCC}}{r} \;\ldots\; \text{Q. E. D.}$$

PROBLEM II.

To determine the value of a given sum, payable if A should be the *second* that fails of the three lives A, B, and C.

SOLUTION.

The sum S may be received in the first year, provided either

* This expression may also be obtained from either of the above general rules, independent of the two others, in like manner as in the solution of the fourth problem.

of three events ſhould happen; 1ſt, if the three lives ſhould become extinct, and A be the ſecond that fails; 2dly, if A ſhould die after B, and C live; and, 3dly, if A ſhould die after C, and B live. But in the ſecond and following years, the given ſum may be received, provided either of ſeven events ſhould happen. 1ſt, If the three lives ſhould fail in the year, A's life having been the ſecond that failed. 2dly, If A only ſhould die in the year, B having died before the beginning and C lived to the end of it. 3dly, If A only ſhould die in the year, C having died before the beginning and B lived to the end of it. 4thly, If A ſhould die after B in the year and C live. 5thly, If A ſhould die after C in the year and B live. 6thly, if A and C ſhould both die in the year (A dying firſt) and B's life ſhould have failed in one or other of the foregoing years; and, 7thly, if A and B ſhould both die in the year (A dying firſt) and C's life ſhould have failed in the foregoing years. From the ſeveral expreſſions denoting theſe contingencies the whole value of the reverſion may be found =

$$\frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c.} + \frac{S}{2ac} + \overline{\frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3} + \&c.} +$$

$$\frac{S}{2ab} \times \overline{\frac{a'b}{r} + \frac{a''m}{r^2} + \frac{a'''n}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c.} -$$

$$\frac{2S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a''ne}{r^3} + \&c.} - \frac{2.S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} -$$

$$\frac{S}{3abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.}$$

The two firſt of theſe ſeries are = D, . . the two next = E . . . and it appears from the ſolution of the preceding problem that the four remaining ſeries expreſs double the value of the ſum S, depending on the contingency of A's dying firſt (when B or C are the oldeſt of the three lives) with a negative ſign.

The

The general rule, therefore, in this case will become $= D + E - 2M$.

But when A is the oldest life, recourse must be had, as in the second part of the preceding problem, to different symbols, and the value of the reversion will then be found $=$

$$\frac{S}{ab} \times \overline{-\frac{ab}{r} + \frac{ms}{r^2} + \frac{nt}{r^3} + \&c.} + \frac{S}{ab} \times \overline{\frac{bs}{r} + \frac{mt}{r^2} + \frac{nu}{r^3} + \&c.} +$$

$$\frac{S}{3abc} \times \overline{\frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c.} +$$

$$\frac{2S}{3abc} \times \overline{\frac{amc'}{r} + \frac{nsc''}{r^2} + \frac{otc'''}{r^3} + \&c.} - \frac{2S}{3abc} \times \overline{\frac{msc'}{r} + \frac{ntc''}{r^2} + \frac{ouc'''}{r^3} + \&c.} +$$

$$\frac{S}{abc} \times \overline{\frac{msc'}{r} + \frac{nt \,.\, \overline{c' + c''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{mtc'}{r} + \frac{nu \,.\, \overline{c' + c''}}{r^2} + \&c.} +$$

$$\frac{S}{abc} \times \overline{\frac{nsc'}{r} + \frac{ot \,.\, \overline{c' + c''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{nt \,.\, c'}{r} + \frac{ou \,.\, \overline{c' + c''}}{r^2} + \&c.} +$$

$$\frac{S}{2ab} \times \overline{\frac{\overline{a-s} \,.\, b}{r} + \frac{\overline{s-t} \,.\, m}{r^2} + \frac{\overline{t-u} \,.\, n}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{\overline{a-s} \,.\, m}{r} + \frac{\overline{s-t} \,.\, n}{r^2} + \&c.} +$$

$$\frac{S}{2ac} \times \overline{\frac{\overline{a-s} \,.\, c}{r} + \frac{\overline{s-t} \,.\, \overline{c-c'}}{r^2} + \&c.} + \frac{S}{2ac} \times \overline{\frac{\overline{a-s} \,.\, \overline{c-c'}}{r} + \frac{\overline{s-t} \,.\, \overline{c-c'+c''}}{r^2} + \&c.} -$$

$$\frac{S}{ab} \times \frac{am}{r} + \frac{ns}{r^2} + \frac{ot}{r^3} + \&c. + \frac{S}{ab} \times \frac{ms}{r} + \frac{nt}{r^2} + \frac{ou}{r^3} + \&c.$$

The last two, and the first ten series are $= -2M$, the eleventh and twelfth series are $= D$, and the thirteenth and fourteenth series $= E$; consequently the general rule becomes $= D + E - 2M$, as before.

Supposing the lives were all equal, the above expression would be $= \frac{S \,.\, \overline{r-1} \,.\, \overline{V - CC}}{r} - \frac{2 \,.\, S}{3r} \times \overline{r-1} \,.\, \overline{V - CCC} = \frac{S}{3} \times \frac{\overline{r-1}}{r} \times \overline{V - 3CC - 2CCC}$, which is known from other principles * to be the true value, and therefore the investigation is right. As a further demonstration, however, it may not be improper to

* See Phil. Transf. Vol. LXXIX.

observe,

observe, that this rule is immediately obtained from the different fractions which express the several contingencies in each year. For in this case, the value of S for the first year becomes $=\frac{S}{r}\times\overline{\frac{1}{3}-\frac{dd}{cc}+\frac{2d^3}{c^3}}$, for the second year $=\frac{S}{r^2}\times\overline{\frac{dd}{cc}-\frac{ee}{cc}-\frac{2d^3}{c^3}+\frac{2e^3}{c^3}}$ and so on for the other years. These series being added together will be found $=\frac{S}{3}\times\overline{\frac{1}{r}+\frac{2d^3}{c^3\,.\,r}+\frac{2e^3}{c^3\,.\,r^2}+\&c.}-\frac{S}{3}\times\overline{\frac{3dd}{ccr}+\frac{3ee}{cc\,.\,r^2}+\&c.}+\frac{S}{3r}\times\overline{\frac{3dd}{ccr}+\frac{3ee}{cc\,.\,r^2}+\&c.}-\frac{S}{3r}\times\overline{\frac{2d^3}{c^3\,.\,r^2}+\frac{2e^3}{c^3\,.\,r^2}+\&c.}=\frac{S}{3}\times\overline{\frac{r-1}{r}}\times\overline{V-3CC-2CCC}$ .... Q. E. D.

PROBLEM III.

To determine the value of a given sum payable on the death of A, if his life should be the *last* that fails of the three lives A, B, C.

SOLUTION.

The given sum can be received in the first year only upon the extinction of the three lives, A having died last. In the second and following years it may be received provided either of four events should happen: 1st, if all the three lives should fail in the year, A dying last; 2dly, if A should die *after* C in the year, B having died in either of the foregoing years; 3dly, if A should die *after* B in the year, C having died in either of the foregoing years; 4thly, if only A should die in the year, B and C having both died before the beginning of it. The value therefore of the reversion (when B or C are older than

than A) will be $= \frac{S}{a} \times \overline{\frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} + \frac{S}{6abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} + \frac{S}{6abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} - \frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c.} - \frac{S}{2ac} \times \overline{\frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{a'b}{r} + \frac{a''m}{r^2} + \frac{a'''n}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c.} = G + M - \overline{D + E}.$

When the life of A is the oldest of the three lives, the symbols being changed as in the two preceding problems, the value of the reversion will become $= -\frac{S}{6abc} \times \overline{\frac{abc'}{r} + \frac{msc''}{r^2} + \frac{ntc'''}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{msc'}{r} + \frac{nt\,.\,\overline{c'+c''}}{r^2} + \&c.} + \frac{S}{6abc} \times \overline{\frac{bsc'}{r} + \frac{mtc''}{r^2} + \frac{nuc'''}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{mt\,.\,c'}{r} + \frac{nu\,.\,\overline{c'+c''}}{r^2} + \&c.} - \frac{S}{3abc} \times \overline{\frac{amc'}{r} + \frac{ns''c}{r^2} + \frac{ot\,.\,c'''}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{ns\,.\,c'}{r} + \frac{ot\,.\,\overline{c'+c'}}{r^2} + \&c.} + \frac{S}{3abc} \times \overline{\frac{msc'}{r} + \frac{nt\,.\,c''}{r^2} + \frac{ou\,.\,c'''}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{nt\,.\,c'}{r} + \frac{ou\,.\,\overline{c'+c''}}{r^2} + \&c.} + \frac{S}{2ac} \times \overline{\frac{\overline{a-s}\,.\,c'}{r} + \frac{\overline{s-t}\,.\,c''}{r^2} + \frac{\overline{t-u}\,.\,c'''}{r^3} + \&c.} + \frac{S}{acr} \times \overline{\frac{\overline{s-t}\,.\,c'}{r} + \frac{\overline{t-u}\,.\,\overline{c'+c''}}{r^2} + \&c.}$

By the second part of the solution of the first problem, the first eight series may be found $= M - E$; and the last two series being easily resolved into $\frac{S}{a} \times \overline{\frac{a-s}{r} + \frac{s-t}{r^2} + \frac{t-u}{r^3} + \&c.} - \frac{S}{2ac} \times \overline{\frac{\overline{a-s}\,.\,\overline{c-c'}}{r} + \frac{\overline{s-t}\,.\,\overline{c-c'+c''}}{r^2} + \&c.} - \frac{S}{2ac} \times \overline{\frac{\overline{a-s}\,.\,c}{r} + \frac{\overline{s-t}\,.\,\overline{c-c'}}{r^2} + \&c.}$ are $= G - D$. Hence the general rule, as in the former case, becomes $= G + M - \overline{D + E}$.

When the lives are all equal, the above expreſſion will be changed into $\frac{S}{r} \times \overline{r-1} \times \overline{V-C+\frac{V-CCC}{3}-V-CC} = \frac{S.\overline{r-1}}{r} \times \frac{\overline{V-L}}{3}$; or "*one-third* the value of the abſolute reverſion after "the extinction of the longeſt of the three lives." This is known from ſelf-evident principles to be the true value, and therefore the foregoing ſolution is correct. The ſame rule may alſo be obtained immediately from the ſeries; for the value in this caſe for the firſt year will be $= \frac{S}{r} \times \overline{\frac{1}{3}-\frac{d}{c}+\frac{dd}{cc}-\frac{d^3}{3c^3}}$, for the ſecond year $= \frac{S}{r^2} \times \overline{\frac{d^3}{3c^3}-\frac{e^3}{3c^3}-\frac{dd}{cc}+\frac{d}{c}+\frac{ee}{cc}-\frac{e}{c}}$, and ſo on for the remaining years. Hence the whole value of the reverſion will be $= \frac{S}{3r} \times L + \frac{S}{3} \times \overline{\frac{1}{r}-L} = \frac{\overline{r-1}.S}{r} \times \frac{\overline{V-L}}{3}$ .... Q. E. D.

PROBLEM IV.

To determine the value of a given ſum payable on the extinction of the lives of A and B, ſhould they be the *firſt* that fail of the three lives A, B, and C.

SOLUTION.

The given ſum may be received in the firſt year, either on the extinction of the three lives, C having died laſt; or on the extinction of only the two lives A and B, C having ſurvived the year. In the ſecond and following years the given ſum may be received, provided either of ſix events ſhould happen. 1ſt, If the three lives ſhould become extinct in the year, C having been the laſt that failed. 2dly, If A only ſhould

ſhould die in the year, B having died before the beginning, and C lived to the end of it. 3dly, If B only ſhould die in the year, A having died before the beginning, and C lived to the end of it. 4thly, If A and B ſhould both die in the year, and C ſurvive it. 5thly, If C ſhould die after A in the year, B having died in either of the foregoing years. 6thly, if C ſhould die after B in the year, A having died in either of the foregoing years. The fractions denoting theſe ſeveral contingencies being added together will be found =

$$\frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''.d}{r^2} + \frac{a'''.e}{r^3} + \&c.} + \frac{S}{2ac} \times \overline{\frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''.f}{r^3} + \&c.} -$$

$$\frac{S}{6abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{md.a'}{r} + \frac{ne.\overline{a'+a''}}{r^2} + \&c.} -$$

$$\frac{S}{3abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{nd.a'}{r} + \frac{oe.\overline{a'+a''}}{r^2} + \&c.} +$$

$$\frac{S}{6abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} + \frac{S}{2abcr} \times \overline{\frac{me.a'}{r} + \frac{nf.\overline{a'+a''}}{r^2} + \&c.} -$$

$$\frac{2S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} - \frac{S}{2abcr} \times \overline{\frac{ne.a'}{r} + \frac{of.\overline{a'+a''}}{r^2} + \&c.}$$

The firſt two ſeries are = D; the third and fourth ſeries are = $\frac{2}{3r} \times \overline{BC - ABC} - \frac{\beta\varkappa.\overline{FK - AFK}}{6.bc}$; the fifth and ſixth ſeries are = $-\frac{\varkappa.\overline{BK - ABK}}{3c} - \frac{m.\overline{PC - APC}}{6br}$; the ſeventh and eighth ſeries = $\frac{\beta.\overline{FC - AFC}}{6b} + \frac{d.\overline{BT - ABT}}{3cr}$; and the ninth and tenth ſeries = $\frac{md.\overline{PT - APT}}{6bcr} - \frac{2.\overline{BC - ABC}}{3}$. Hence the whole value of the reverſion will be = $D - \frac{S.\varkappa}{3c} \times \overline{\frac{\beta.\overline{FK - AFK}}{2b} + \overline{BK - ABK}} + \frac{S.\beta}{6b} \times \overline{FC - AFC} - \frac{2.S.\overline{r-1}}{3r} \times \overline{BC - ABC} - \frac{S.m}{6br} \times \overline{PC - APC} + \frac{S.d}{3cr} \times \overline{\overline{BT - ABT} + \frac{m.\overline{PT - APT}}{2b}}$.

The preceding theorem expreſſes the value of the given ſum whether C be the oldeſt, or B one of the two lives to be ſurvived, and will therefore be ſufficient in all caſes. In order, however, ſtill more fully to prove this, let A be ſuppoſed the oldeſt life, and inſtead of $\overline{b-m}$, $\overline{m-n}$, $\overline{n-o}$, &c. and a', a'', a''', &c. let b', b'', b''', &c. and $\overline{a-s}$, $\overline{s-t}$, $\overline{t-u}$, &c. be ſubſtituted; then will the value of the reverſion be found =

$$\frac{S}{2bc}\times\overline{\frac{cb'}{r}+\frac{db''}{r^2}+\frac{eb'''}{r^3}+\&c.}+\frac{S}{2bc}\times\overline{\frac{ab'}{r}+\frac{eb''}{r^2}+\frac{fb'''}{r^3}+\&c.}-$$

$$\frac{S}{6abc}\times\overline{\frac{acb'}{r}+\frac{dsb''}{r^2}+\frac{etb'''}{r^3}+\&c.}+\frac{S}{2abcr}\times\overline{\frac{db}{r}+\frac{t\,.\,\overline{b'+b''}}{r^2}+\&c.}-$$

$$\frac{S}{3abc}\times\overline{\frac{csb'}{r}+\frac{dtb''}{r^2}+\frac{eub'''}{r^3}+\&c.}-\frac{S}{2abcr}\times\overline{\frac{db'}{r}+\frac{eu\,.\,\overline{b'+b''}}{r^2}+\&c.}+$$

$$\frac{S}{6abc}\times\overline{\frac{adb'}{r}+\frac{seb''}{r^2}+\frac{ftb'''}{r^3}+\&c.}+\frac{S}{2abcr}\times\overline{\frac{seb'}{r}+\frac{tf\,.\,\overline{b'+b''}}{r^2}+\&c.}-$$

$$\frac{2S}{3abc}\times\overline{\frac{dsb'}{r}+\frac{et\,.\,b''}{r^2}+\frac{fu\,.\,b'''}{r^3}+\&c.}-\frac{S}{2abcr}\times\overline{\frac{et\,.\,b'}{r}+\frac{fu\,.\,\overline{b'+b''}}{r^2}+\&c.}$$

Let Σ denote the value of S on the contingency of C's ſurviving B, and the general rule deduced from the preceding ſeries will become $=\Sigma-\frac{S\,.\,x}{3}\times\overline{\frac{\alpha\,.\,\overline{HK-HBK}}{2a}+\overline{AK-ABK}}+\frac{S\,.\,x}{6a}\times\overline{HC-HBC}-\frac{2S\,.\,\overline{r-1}}{3r}\times\overline{AC-ABC}-\frac{S\,.\,s}{6ar}\times\overline{NC-NBC}+\frac{S\,.\,d}{3cr}\times\overline{\overline{AT-ABT}+\frac{s\,.\,\overline{NT-NBT}}{2a}}$, which appears to be exactly the ſame rule with the foregoing, if the ſymbols of A and B be only exchanged for each other.

If the three lives be of the ſame age, both thoſe rules will ſeverally become = S into $\frac{\overline{r-1}}{2r}\times\overline{V-CC}-\frac{xx}{6cc}\times\overline{KK-CKK}-\frac{x}{6c}\times\overline{CK-CCK}-\frac{2\,.\,\overline{r-1}}{3r}\times\overline{CC-CCC}+\frac{d}{6cr}\times\overline{CT-CCT}+\frac{dd}{6ccr}\times$

$\overline{TT-CTT}$. The two expressions $-\frac{xx\,.\,\overline{KK-CKK}}{6cc}+\frac{d}{6cr}\times$ $\overline{CT-CCT}$, by resolving them into their respective series, will be found $=\frac{1-CC}{6r}+\frac{d}{6cr}+\frac{de}{6c^2r^2}+\frac{ef}{6c^2r^3}+\frac{fg}{6c^2r^4}+$ &c., and the two expressions $\frac{dd\,.\,\overline{TT-CTT}}{6ccr}-\frac{x}{c}\times\overline{CK-CCK}=\frac{CC}{6r}-\frac{d}{6cr}-\frac{de}{6ccr^2}+\frac{ef}{6ccr^3}+\frac{fg}{6ccr^4}+$ &c.; hence the sum of these four expressions will be $=-\frac{\overline{r-1}\,.\,\overline{V-CC}}{6r}$, and consequently the general rule in this case will be $=\frac{S\,.\,\overline{r-1}}{3r}\times\overline{V-3CC-2CCC}$, which is known to denote the true value from other principles *.

As a further proof of the accuracy of the preceding investigation, it may not be improper to observe, that this rule may be immediately obtained from the different fractions which express the value of the given sum in each year. For in this case the value of S in the first year is $=\frac{S}{3}\times\overline{\frac{1}{r}-\frac{3dd}{ccr}+\frac{2d^3}{c^3\,.\,r}}$, in the second year $=\frac{S}{3}\times\overline{\frac{2c^3}{c^3r^2}-\frac{3ee}{ccr^2}-\frac{2d^3}{c^3r^2}+\frac{3dd}{ccr^2}}$, and so on in the other years, which expressions may be easily found $=$ $\frac{S\,.\,\overline{r-1}}{3r}\times\overline{V-3CC-2CCC}$. Q. E. D.

PROBLEM V.

To find the value of a given sum, payable on the death of A, if his life should be the *first* or *second* that fails of the three lives A, B, and C.

* See my Paper in the Phil. Transf. Vol. LXXIX.

SOLU-

SOLUTION.

In the first year the payment of the given sum depends upon either of four events. 1st, That all the three lives shall become extinct, the life of A having been the first or second that failed. 2dly, That A and B shall both die, and C live to the end of the year. 3dly, That A and C shall both die, and B live to the end of the year. 4thly, That only A shall die, and B and C both live to the end of the year. In the second and following years the payment of the given sum will depend upon either of eight events happening. 1st, That all the three lives shall drop in the year, A having been the first or second that failed. 2dly, That C survives, and only A and B die in the year. 3dly, That B survives, and only A and C die in the year. 4thly, That both B and C survive, and A only dies in the year. 5thly, That A dies in the year, B having died before the beginning, and C lived to the end of it. 6thly, That A in like manner dies in the year, C having died before the beginning, and B lived to the end of it. 7thly, That C dies after A in the year, B's life having failed in either of the preceding years. 8thly, That B dies after A in the year, C's life having failed in either of the preceding years. The fractions denoting these several contingencies are

$$= \frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''.d}{r^2} + \frac{a'''.e}{r^3} + \&c.} + \frac{S}{2ac} \times \overline{\frac{a'd}{r} + \frac{a''.e}{r^2} + \frac{a'''f}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{a'b}{r} + \frac{a''.m}{r^2} + \frac{a'''n}{r^3} + \&c.} + \frac{S}{2ab} \times \overline{\frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'm}{r} + \frac{a''nd}{r^2} + \frac{a''oe}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''n}{r^2} + \frac{a'''of}{r^3} + \&c.} = D + E - M.$$

This

This general rule gives the true value whether the life of A be older or younger than both or either of the lives of B and C. When the three lives are of equal age, the value of S for the firſt year will be $= \frac{S}{r} \times \overline{\frac{2}{3} + \frac{d^3}{c^3} - \frac{dd}{cc}}$, for the ſecond year $= \frac{S}{r^2} \times \overline{\frac{e^3}{3c^3} - \frac{ee}{cc} - \frac{d^3}{3c^3} + \frac{dd}{cc}}$, and ſo on for the other years. Hence the whole value in this caſe will be $= \frac{S . \overline{r-1}}{3r} \times \overline{2V - \overline{3CC - CCC}}$, which expreſſion may alſo be derived from the general rule juſt given above, or $D + E - M$.

The ſolution of this problem may alſo be obtained either from the firſt and ſecond, or from the third problems. In the one caſe the value of S is evidently equal to the *ſum* of the two values determined by the two firſt-mentioned problems, or $D + E - 2M + M = D + E - M$. And in the other caſe its value is equal to the *difference* between the abſolute value of the reverſion after A ($= G$) and its value depending upon the contingency that A ſhall be the *laſt* life that ſhall fail, which being $= G + M - \overline{D + E}$ by the third problem, it follows, that the general rule on this ſuppoſition will be alſo $= D + E - M$. Q. E. D.

PROBLEM VI.

To find the value of a given ſum payable on the death of A, ſhould his life be the *ſecond* or *third* that ſhall fail of the three lives A, B, and C.

SOLUTION.

The payment of the given ſum in the firſt year will depend upon the contingency of either of three events. 1ſt, That all the three lives shall become extinct, A having been the second or third that has failed. 2dly, That A shall die after B, and C live to the end of the year. 3dly, That A shall die after C, and B live to the end of the year. In the second and following years the given sum will become payable, provided either of eight events should happen. 1st, If all the three lives should fail in the year, A having been the second or third that died. 2dly, If A should die after B in the year, and C live to the end of it. 3dly, If A should die after C in the year, and B live to the end of it. 4thly, If A and C should both die in the year, B having died before the beginning of it. 5thly, If A and B should both die in the year, C having died before the beginning of it. 6thly, If A only should die in the year, the lives of B and C having become extinct in either of the preceding years. 7thly, If A should die in the year, B having died before the beginning, and C lived to the end of it. 8thly, if A should die in the year, C having died before the beginning, and B lived to the end of it. Hence the whole value of the reversion will be found =

$$\frac{S}{a} \times \overline{\frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} - \frac{S}{6abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} = G - M.$$

This rule is correct in all cases; but when the three lives are equal it becomes more simple, and is $= \frac{S \cdot \overline{r-1}}{3r} \times \overline{2V - 3C - \overline{CCC}}$; which expression may likewise be obtained immediately from the series given above.

The solution of this problem, like that of the foregoing one, may also be derived from the first three problems; for the value

value of S is either equal to the *difference* between the abſolute value of the reverſion after the death of A and its value depending on the contingency that A ſhall be the *firſt* that fails (found by Prob. 1.), or it is equal to the *ſum* of the two values depending on the contingencies that A ſhall be the *ſecond* or *third* that fails (found by Prob. 2. and 3.). In both caſes the general rule is $= G - M$. Q. E. D.

PROBLEM VII.

To find the value of a given ſum payable on the death of A, ſhould his life be the *firſt* or *laſt* that fails of the three lives A, B, C.

SOLUTION.

In order to receive the given ſum in the firſt year, it is neceſſary that either of four events ſhould happen. 1ſt, That all the three lives ſhould fail, A having been the firſt or third that died. 2dly, That B ſhould die after A, and C live. 3dly, That C ſhould die after A, and B live. 4thly, That A only ſhould die, and B and C both live. In the ſecond and following years the given ſum may be received, provided either of ſeven events ſhould happen. 1ſt, If the three lives ſhould fail, A having been the firſt or laſt that died. 2dly, if B ſhould die after A in the year, and C live to the end of it. 3dly, If C ſhould die after A in the year, and B live to the end of it. 4thly, If A only ſhould die in the year, and B and C both live to the end of it. 5thly, If A's life ſhould fail after that of B in the year, C's life having failed before the beginning of it. 6thly, If A ſhould fail after C in the year, B having failed before the beginning of it. 7thly, If A only ſhould

ſhould die in the year, B and C having died in either of the preceding years. From the fractions denoting theſe ſeveral contingencies the whole value of the reverſion will be found =

$$\frac{S}{a} \times \overline{\frac{a'}{r} + \frac{a''}{r^2} + \frac{a'''}{r^3} + \&c.} - \frac{S}{2ac} \times \overline{\frac{a'c}{r} + \frac{a''d}{r^2} + \frac{a'''e}{r^3} + \&c.} -$$
$$\frac{S}{2ac} \times \overline{\frac{a'd}{r} + \frac{a''e}{r^2} + \frac{a'''f}{r^3} + \&c.} - \frac{S}{2ab} \times \overline{\frac{a'b}{r} + \frac{a''m}{r^2} + \frac{a'''n}{r^3} + \&c.} -$$
$$\frac{S}{2ab} \times \overline{\frac{a'm}{r} + \frac{a''n}{r^2} + \frac{a'''o}{r^3} + \&c.} + \frac{2S}{3abc} \times \overline{\frac{a'bc}{r} + \frac{a''md}{r^2} + \frac{a'''ne}{r^3} + \&c.} +$$
$$\frac{S}{3abc} \times \overline{\frac{a'mc}{r} + \frac{a''nd}{r^2} + \frac{a'''oe}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{a'bd}{r} + \frac{a''me}{r^2} + \frac{a'''nf}{r^3} + \&c.} +$$
$$\frac{2S}{3abc} \times \overline{\frac{a'md}{r} + \frac{a''ne}{r^2} + \frac{a'''of}{r^3} + \&c.} = G - \overline{D + E} + 2M.$$

This general theorem will give the exact value in all caſes; but when the lives are equal, it is rendered more ſimple, by ſubſtituting the ſeveral values of G, D, E, and M, and will then become $= \frac{S \cdot \overline{r-1}}{3r} \times \overline{2V - 3C - 3CC + 2CCC}$; which expreſſion may alſo be derived in this particular caſe from the different ſeries denoting the value of S in each year.

The ſolution of this, like thoſe of the two preceding problems, may likewiſe be obtained by the aſſiſtance of the firſt three problems. For the value of this contingent reverſion is either equal to the *ſum* of the two values of S payable on the death of A, if his life ſhould be the firſt, or if it ſhould be the laſt that fails (found by Prob. 1. and 3.), or it is equal to the *difference* between the value of the abſolute reverſion after A's death, and the value of the contingent reverſion after A's death, provided he ſhould be the *ſecond* that fails of the three lives (found by Prob. 2.). In both caſes the general rule becomes $= G - \overline{D + E} + 2M$. Q. E. D.

PRO

PROBLEM VIII.

To find the value of a given ſum payable on the death of A or B, ſhould *either* of them be the *firſt* that ſhall fail of the three lives A, B, and C.

SOLUTION.

In each year the payment of the given ſum will depend upon either of ſix events. 1ſt, If the three lives ſhould fail in the year, A or B having died firſt. 2dly, if A and B ſhould die in the year, and A live. 3dly, If C ſhould die after A in the year, and B live. 4thly, If C ſhould die after B in the year, and A live. 5thly, If A only ſhould die in the year, and B and C both live. 6thly, If B only ſhould die in the year, and A and C both live. The fractions denoting theſe ſeveral contingencies are

$$= \frac{2S}{3abc}\times\overline{\frac{bca'}{r}+\frac{md\cdot a''}{r^2}+\frac{nea'''}{r^3}+\&c.} - \frac{S}{6abc}\times\overline{\frac{mca'}{r}+\frac{nda''}{r^2}+\frac{oea'''}{r^3}+\&c.} + \frac{S}{3abc}\times\overline{\frac{bda'}{r}+\frac{mea''}{r^2}+\frac{nf\,.\,a'''}{r^3}+\&c.} + \frac{S}{6abc}\times\overline{\frac{mda'}{a}+\frac{nea''}{r^2}+\frac{ofa'''}{r^3}+\&c.} + \frac{S}{2bc}\times\overline{\frac{\overline{b-m}\,.\,c}{r}+\frac{\overline{m-n}\,.\,d}{r^2}+\frac{\overline{n-o}\,.\,e}{r^3}+\&c.} + \frac{S}{2bc}\times\overline{\frac{\overline{b-m}\,.\,d}{r}+\frac{\overline{m-n}\,.\,e}{r^2}+\frac{\overline{n-o}\,.\,f}{r^3}+\&c.} - \frac{S}{2abc}\times\overline{\frac{bca'}{r}+\frac{md\,.\,\overline{a'+a''}}{r^2}+\&c.} + \frac{S}{2abc}\times\overline{\frac{mca'}{r}+\frac{nd\,.\,\overline{a'+a'}}{r^2}+\&c.} - \frac{S}{2abc}\times\overline{\frac{bda'}{r}+\frac{me\,.\,\overline{a'+a''}}{r^2}+\&c.} + \frac{S}{2abc}\times\overline{\frac{md\,.\,a'}{r}+\frac{ne\,.\,\overline{a'+a''}}{r^2}+\&c.} = S \text{ into } \frac{\kappa}{3c}\times\overline{\frac{\beta\,.\,\overline{FK-AFK}}{2b}+\overline{BK-ABK}-\frac{\beta\,.\,\overline{FC-AFC}}{6b}+\frac{2\,.\,\overline{r-1}\,.\,\overline{BC-ABC}}{3r}+\frac{m\,.\,\overline{PC-APC}}{6br}} - \frac{d}{3cr}\times\overline{\overline{BT-ABT}+\frac{m\,.\,\overline{PT-APT}}{2b}}+\Sigma$$

(Σ denoting the value of S on the contingency of C's ſurviving B, as in Prob. 4.)

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The above rule gives the exact value when C is the oldest of the three lives. But if A be the oldest, the symbols must be changed as in some of the foregoing problems, and the value in this case will be expressed by the series $\frac{2S}{3abc}\times\overline{\frac{abc'}{r}+\frac{msc''}{r^2}+\frac{ntc'''}{r^3}+\&c.}-\frac{S}{6abc}\times\overline{\frac{amc'}{r}+\frac{nsc''}{r^2}+\frac{otc'''}{r^3}+\&c.}-\frac{S}{6abc}\times\overline{\frac{bsc'}{r}+\frac{mtc''}{r^2}+\frac{nuc'''}{r^3}+\&c.}-\frac{S}{3abc}\times\overline{\frac{msc'}{r}+\frac{ntc''}{r^2}+\frac{ouc'''}{r^3}+\&c.}+\frac{S}{ab}\times\overline{\frac{ab}{r}+\frac{ms}{r^2}+\frac{nt}{r^3}+\&c.}-\frac{S}{ab}\times\overline{\frac{ms}{r}+\frac{nt}{r^2}+\frac{ou}{r^3}+\&c.}-\frac{S}{abc}\times\overline{\frac{abc'}{r}+\frac{ms\,.\,\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{abc}\times\overline{\frac{msc'}{r}+\frac{nt\,.\,\overline{c'+c''}}{r^2}+\&c.}$ From these several series, the general rule expressing the value of the reversion will be found $=$ S into $\frac{\overline{r-1}\,.\,\overline{V-AB}}{r}-\frac{\alpha}{3a}\times\overline{\frac{\beta\,.\,\overline{HF-HFC}}{b}+\frac{HB-HBC}{2}-\frac{\beta\,.\,\overline{AF-AFC}}{6b}+\frac{2\,.\,\overline{r-1}\,.\,\overline{AB-ABC}}{3r}+\frac{m\,.\,\overline{AP-APC}}{6br}}+\frac{S}{3ar}\times\overline{\frac{BN-BNC}{2}+\frac{m\,.\,\overline{PN-PNC}}{b}}$.

When the lives are all equal, the first rule becomes $=\frac{\overline{r-1}\,.\,\overline{V-CC}}{2r}+\frac{2\,.\,\overline{r-1}\,.\,\overline{CC-CCC}}{3r}+\frac{\kappa\kappa\,.\,\overline{KK-CKK}}{6cc}+\frac{\kappa\,.\,\overline{CK-CCK}}{6c}-\frac{d\,.\,\overline{CT-CCT}}{6cr}-\frac{dd\,.\,\overline{TT-CTT}}{6ccr}$, and the second rule $=\frac{\overline{r-1}\,.\,\overline{V-CC}}{r}+\frac{2\,.\,\overline{r-1}\,.\,\overline{CC-CCC}}{3r}-\frac{\kappa\kappa\,.\,\overline{KK-CKK}}{3cc}-\frac{\kappa\,.\,\overline{KC-CCK}}{3c}+\frac{d\,.\,\overline{CT-CCT}}{3cr}+\frac{dd\,.\,\overline{TT-CTT}}{3ccr}$. In the one case the four last fractions are $=\frac{\overline{r-1}\,.\,\overline{V-CC}}{6r}$; and in the other case those fractions are $=-\frac{\overline{r-1}\,.\,\overline{V-CC}}{3r}$; therefore, in both cases the general rule becomes $=\frac{2\,.\,S\,.\,\overline{r-1}}{3r}\times\overline{V-CCC}$, which is known to be the true value from

from ſelf-evident principles. This expreſſion may alſo be immediately derived from the different fractions which denote the value of S in each year. For in the firſt year theſe fractions will be reduced to $\frac{2S}{3} \times \overline{\frac{1}{r} - \frac{d^3}{c^3 r}}$, in the ſecond year to $\frac{2S}{3} \times \overline{\frac{d^3 - e^3}{c^3 r^2}}$, in the third year to $\frac{2S}{3} \times \overline{\frac{e^3}{c^3 r^3} - \frac{f^3}{c^3 r^3}}$, and ſo on in the other years. Hence the whole value of the reverſion will, as above, be $= \frac{2 . S . \overline{r-1}}{3r} \times \overline{V - CCC}$. Q. E. D *.

PROBLEM IX.

To determine the value of a given ſum payable on the death of A or B, ſhould *either* of them be the *ſecond* that fails of the three lives A, B, and C.

SOLUTION.

The payment of the given ſum in the firſt year will depend upon either of four events happening. 1ſt, That the three lives ſhould fail, A or B having been the ſecond that failed. 2dly, That A ſhould die after C, and B live. 3dly, That B ſhould die after C, and A live. 4thly, That A and B ſhould both die, and C live. In the ſecond and following years the given ſum will become payable, provided either of eleven events ſhould happen. 1ſt, If the three lives ſhould drop in the year, A or B having been the ſecond that failed. 2dly, If A ſhould die after C in the year, and B live. 3dly, If B ſhould die after C in the year, and A live. 4thly, If A and B

* I do not know that any ſolution has been attempted before, either of this or of the two following problems.

ſhould

ſhould both die in the year, and C live. 5thly, if B only ſhould die in the year, A having died before the beginning, and C lived to the end of it. 6thly, If A only ſhould die in the year, B having died before the beginning, and C lived to the end of it. 7thly, If C ſhould die after A in the year, B having died in either of the foregoing years. 8thly, If C ſhould die after B in the year, A having died in either of the foregoing years. 9thly, If A and B ſhould both die in the year, C having died in either of the preceding years. 10thly, If B only ſhould die in the year, C having died before the beginning, and A lived to the end of it. And, laſtly, if A only ſhould die in the year, C having died before the beginning, and B lived to the end of it. The ſeveral fractions denoting theſe contingencies in each year being added together will become

$$= \frac{S}{abc} \times \overline{\frac{bca'}{r} + \frac{md\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{mca'}{r} + \frac{nd\,.\,\overline{a'+a''}}{r^2} + \&c.}$$

$$+ \frac{S}{abc} \times \overline{\frac{bda'}{r} + \frac{me\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{abc} \times \overline{\frac{mda'}{r} + \frac{ne\,.\,\overline{a'+a''}}{r^2} + \&c.} -$$

$$\frac{4S}{3abc} \times \overline{\frac{bca'}{r} + \frac{mda''}{r^2} + \frac{nea'''}{r^3} + \&c.} + \frac{S}{3abc} \times \overline{\frac{mca'}{r} + \frac{nda''}{r^2} + \frac{oea''}{r^3} + \&c.} -$$

$$\frac{2S}{3abc} \times \overline{\frac{bda'}{r} + \frac{mea''}{r^2} + \frac{nfa'''}{r^3} + \&c.} - \frac{S}{3abc} \times \overline{\frac{mda'}{r} + \frac{nea''}{r^2} + \frac{of\,.\,a'''}{r^3} + \&c.} -$$

$$\frac{S}{2bc} \times \overline{\frac{\overline{b-m}\,.\,c}{r} + \frac{\overline{m-n}\,.\,d}{r^2} + \frac{\overline{n-o}\,.\,e}{r^3} + \&c.} - \frac{S}{2bc} \times \overline{\frac{\overline{b-m}\,.\,d}{r} + \frac{\overline{m-n}\,.\,e}{r^2} + \frac{\overline{n-o}\,.\,f}{r^3}}$$

$$\overline{+ \&c.} + \frac{S}{ab} \times \overline{\frac{ma'}{r} + \frac{n\,.\,\overline{a'+a''}}{r^2} + \&c.} - \frac{S}{ab} \times \overline{\frac{ba'}{r} + \frac{m\,.\,\overline{a'+a''}}{r^2} + \&c.} +$$

$$\frac{S}{ab} \times \overline{\frac{ba'}{r} + \frac{ma''}{r^2} + \frac{na''}{r^3} + \&c.} + \frac{S}{b} \times \overline{\frac{\overline{b-m}\,.}{r} + \frac{\overline{m-n}\,.}{r^2} + \frac{\overline{n-o}}{r^3} + \&c.} +$$

$$\frac{S}{2ac} \times \overline{\frac{ca'}{r} + \frac{da''}{r^2} + \frac{ea'''}{r^3} + \&c.} + \frac{S}{2ac} \times \overline{\frac{da'}{r} + \frac{ea''}{r^2} + \frac{fa'''}{r^3} + \&c.}$$

Let Q denote the value of S by the firſt rule in the eighth problem, and the firſt ten ſeries will be $= \Sigma - 2Q$. The four next ſeries

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are $=\frac{S.\overline{r-1}.\overline{V-AB}}{r}$; and the two laſt ſeries are $=D$; the whole value of the reverſion, therefore (when C is the oldeſt of the three lives), is $=\frac{S.\overline{r-1}.\overline{V-AB}}{r}+D+\Sigma-2Q$.

When A is the oldeſt of the three lives, the different ſeries, by changing the ſymbols, as in ſome of the foregoing problems, will become $=\frac{2S}{abc}\times\overline{\frac{abc'}{r}+\frac{ms.\overline{c'+c''}}{r^2}+\&c.}-\frac{2S}{abc}\times\overline{\frac{msc'}{r}+\frac{nt.\overline{c'+c''}}{r^2}+\&c.}-$ $\frac{4S}{3abc}\times\overline{\frac{abc'}{r}+\frac{msc''}{r^2}+\frac{nt.c'''}{r^3}+\&c.}+\frac{2S}{3abc}\times\overline{\frac{ms.c'}{r}+\frac{nt.c''}{r^2}+\frac{ou.c'''}{r^3}+\&c.}$ $+\frac{S}{3abc}\times\overline{\frac{amc'}{r}+\frac{nsc''}{r^2}+\frac{otc'''}{r^3}+\&c.}+\frac{S}{3abc}\times\overline{\frac{bsc'}{r}+\frac{mtc''}{r^2}+\frac{nuc'''}{r^3}+\&c.}-$ $\frac{S}{bc}\times\overline{\frac{bc'}{r}+\frac{m.\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{bc}\times\overline{\frac{mc'}{r}+\frac{n.\overline{c'+c''}}{r^2}+\&c.}+$ $\frac{S}{2abc}\times\overline{\frac{bc'}{r}+\frac{mc''}{r^2}+\frac{nc'''}{r^3}+\&c.}-\frac{S}{2bc}\times\overline{\frac{mc'}{r}+\frac{nc''}{r^2}+\frac{oc''}{r^3}+\&c.}-$ $\frac{S}{ac}\times\overline{\frac{ac'}{r}+\frac{s.\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{ac}\times\overline{\frac{sc'}{r}+\frac{t.\overline{c'+c''}}{r^2}+\&c.}+$ $\frac{S}{2ac}\times\overline{\frac{ac'}{r}+\frac{sc''}{r^2}+\frac{tc'''}{r^3}+\&c.}-\frac{S}{2ac}\times\overline{\frac{sc'}{r}+\frac{tc''}{r^2}+\frac{uc'''}{r^3}+\&c.}+$ $\frac{S}{b}\times\overline{\frac{\overline{b-m}}{r}+\frac{\overline{m-n}}{r^2}+\frac{\overline{n-o}}{r^3}+\&c.}-\frac{S}{ab}\times\overline{\frac{\overline{b-m}.s}{r}+\frac{\overline{m-n}.t}{r^2}+\frac{\overline{n-o}.u}{r^3}+\&c.}$ $+\frac{S}{abr}\times\overline{\frac{\overline{b-m}.s}{r}+\frac{\overline{m-n}.t}{r^2}+\&c.}$ Let Q repreſent the value of S by the ſecond rule in the eighth problem, Π and Δ the values of the ſame ſum on the contingency of B's dying after C, and on the contingency of A's dying after C reſpectively *; then will the general rule in this caſe become $=\frac{S.\overline{r-1}.\overline{3V-B-A-AB}}{r}$ $-2Q-\overline{\Pi+\Delta}$, or, becauſe $D+\Sigma$ are $=\frac{\overline{r-1}.\overline{2V-B-A}}{r}-\overline{\Pi+\Delta}$ †,

* See my 3d Prob. in Phil. Tranſ. Vol. LXXVIII.

† See my 2d and 3d Prob. Vol. LXXVIII.

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it will be $= \frac{S \,.\, \overline{r-1} \,.\, \overline{V-AB}}{r} + D + \Sigma - 2Q$, as above.

When the lives are equal, the rule may be found either immediately from the series, or from the foregoing expressions, $= \frac{2S \,.\, \overline{r-1}}{3r} \times \overline{V - 3CC - 2CCC}$. Q. E. D.

PROBLEM X.

To find the value of a given sum payable on the decease of B or C, should *either* of them be the *last* that fails of the three lives A, B, and C.

SOLUTION.

The sum S can be received in the first year only on the extinction of the three lives, restrained to the contingency of A's life having been the first or second that failed. In the second and following years it may be received provided either of six events should happen. 1st, If the three lives should fail in the year, A having been the first or second that died. 2dly, If B and C should both die in the year, A having died before the beginning of it. 3dly, If C only should die in the year, A and B having died in either of the preceding years. 4thly, If B only should die in the year, A and C having died in either of the preceding years. 5thly, If B should die after A in the year, C having died before the beginning of it. 6thly, If C should die after A in the year, B having died before the beginning of it. From the fractions expressing these several contingencies, the whole value of the reversion will be found $=$

$$\frac{2S}{3abc} \times \overline{\frac{mda'}{r} + \frac{ne \,.\, \overline{a'+a''}}{r^2} + \&c.} - \frac{S}{3abc} \times \overline{\frac{bca'}{r} + \frac{md \,.\, \overline{a'+a''}}{r^2} + \&c.} - \frac{S}{6abc}$$

$\frac{S}{6abc}\times\overline{\frac{mca'}{r}+\frac{nda''}{r^2}+\frac{oea'''}{r^3}+\&c.}-\frac{S}{6abc}\times\overline{\frac{bda'}{r}+\frac{mea''}{r^2}+\frac{nf.a'''}{r^3}+\&c.}+$ $\frac{S}{2ab}\times\overline{\frac{ba'}{r}+\frac{m.\overline{a'+a''}}{r^2}+\frac{n.\overline{a'+a''+a'''}}{r^3}+\&c.}-\frac{S}{2ab}\times\overline{\frac{ma'}{r}+\frac{n.\overline{a'+a''}}{r^2}+\&c.}+\frac{S}{2ac}\times\overline{\frac{ca'}{r}+\frac{d.\overline{a'+a''}}{r^2}+\&c.}-\frac{S}{2ac}\times\overline{\frac{da'}{r}+\frac{e.\overline{a'+a''}}{r^2}+\&c.}-$ $\frac{2S}{3abcr}\times\overline{\frac{mda'}{r}+\frac{ne.\overline{a'+a''}}{r^2}+\&c.}+\frac{S}{3abcr}\times\overline{\frac{nea'}{r}+\frac{of.\overline{a'+a''}}{r^2}+\&c.}+$ $\frac{S}{2abr}\times\overline{\frac{ma'}{r}+\frac{n.\overline{a'+a''}}{r^2}+\&c.}-\frac{S}{2abr}\times\overline{\frac{na'}{r}+\frac{o.\overline{a'+a''}}{r^2}+\&c.}+$ $\frac{S}{2acr}\times\overline{\frac{d.a'}{r}+\frac{e.\overline{a'+a''}}{r^2}+\&c.}-\frac{S}{2acr}\times\overline{\frac{ea'}{r}+\frac{f.\overline{a'+a''}}{r^2}+\&c.}$, and therefore the general rule (when either B or C are the oldest of the three lives) will be = S into $\frac{2.\overline{r-1}.\overline{BC-ABC}}{3r}-\frac{\varkappa}{3c}\times\overline{\frac{\beta.\overline{FK-AFK}}{b}+\frac{\overline{BK-ABK}}{2}}\times\frac{m}{3br}\times\overline{\frac{\overline{PC-APC}}{2}+\frac{d.\overline{PT-APT}}{c}}-\frac{\beta.\overline{FC-AFC}}{6b}$ $+\frac{d.\overline{BT-ABT}}{6cr}+\frac{\beta.\overline{F-AF}}{2b}-\frac{\overline{r-1}.\overline{B-AB}}{2r}-\frac{m.\overline{P-AP}}{2br}+\frac{\varkappa.\overline{K-AK}}{2c}-$ $\frac{\overline{r-1}.\overline{C-AC}}{2r}-\frac{d.\overline{T-AT}}{2cr}$; or (since the first five expressions are = $\frac{\overline{r-1}.\overline{BC-ABC}}{r}-M$, and the six remaining ones $=R+W$) it will be $=\frac{S.\overline{r-1}.\overline{BC-ABC}}{r}+R+W-M$.

When A *is the oldest of the three lives*, the same general rule may be obtained. In this case, by exchanging the symbols, the different series will become $=\frac{S}{6abc}\times\overline{\frac{abc'}{r}+\frac{ms.\overline{c'+c''}}{r^2}+\&c.}-$ $\frac{S}{6abc}\times\overline{\frac{bs.c'}{r}+\frac{mt.\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{3abc}\times\overline{\frac{amc'}{r}+\frac{ns.\overline{c'+c''}}{r^2}+\&c.}+$ $\frac{2S}{3abc}\times\overline{\frac{ms.c'}{r}+\frac{nt.\overline{c'+c''}}{r^2}+\&c.}+\frac{S}{c}\times\overline{\frac{c'}{r}+\frac{c''}{r^2}+\frac{c'''}{r^3}+\&c.}-$

$\frac{S}{bc} \times \overline{\frac{mc'}{r} + \frac{n \,.\, \overline{c'+c'}}{r^2} + \&c.} - \frac{S}{2ac} \times \overline{\frac{ac'}{r} + \frac{sc''}{r^2} + \frac{tc'''}{r^3} + \&c.} -$
$\frac{S}{2ac} \times \overline{\frac{sc'}{r} + \frac{t \,.\, c''}{r^2} + \frac{uc'''}{r^3} + \&c.} - \frac{2S}{3abcr} \times \overline{\frac{ms \,.\, c'}{r} + \frac{nt \,.\, \overline{c+c''}}{r^2} + \&c.} -$
$\frac{S}{3abcr} \times \overline{\frac{mt \,.\, c'}{r} + \frac{ru \,.\, \overline{c'+c''}}{r^2} + \&c.} + \frac{S}{6abcr} \times \overline{\frac{rsc'}{r} + \frac{ot \,.\, \overline{c'+c''}}{r^2} + \&c.} -$
$\frac{S}{6abcr} \times \overline{\frac{nt \,.\, c'}{r} + \frac{ou \,.\, \overline{c'+c''}}{r^2} + \&c.} + \frac{S}{bcr} \times \overline{\frac{mc'}{r} + \frac{n \,.\, \overline{c'+c''}}{r^2} + \&c.} =$ S into
$\frac{2 \,.\, \overline{r-1} \,.\, \overline{AB-ABC}}{3^r} + \frac{\beta}{6b} \times \overline{\frac{\alpha \,.\, \overline{HF-HFC}}{a} - \overline{AF-AFC}} + \frac{\alpha \,.\, \overline{HB-HBC}}{3a}$
$- \frac{s}{3ar} \times \overline{NB-NBC} + \frac{m \,.\, \overline{PN-PNC}}{2b} + \frac{m \,.\, \overline{AP-APC}}{6br} + \frac{\overline{r-1} \,.\, \overline{V-C}}{r} -$
$\frac{\overline{r-1} \,.\, \overline{B-BC}}{r} - \frac{\overline{r-1} \,.\, \overline{A-AC}}{2r} - \frac{\alpha \,.\, \overline{H-HC}}{2a} + \frac{s \,.\, \overline{N-NC}}{2ar}$. In order to get the ſame general rule as that given above in the caſe of B or C's being the oldeſt of the three lives, it is to be obſerved, that the firſt five expreſſions, by the ſecond part of the ſolution of Prob. 1. appear to be $= \frac{\overline{r-1} \,.\, \overline{AB-ABC}}{r} - M + E$. And (ſuppoſing Γ to repreſent the value of S on the contingency of A's dying after C) that the laſt three expreſſions appear to be $= -\Gamma - \frac{\overline{r-1} \,.\, \overline{A-AC}}{r}$. But E is $= R + \frac{\overline{r-1} \,.\, \overline{B-AB}}{r}$, and W muſt be $= \frac{\overline{r-1} \,.\, \overline{V-A-C+AC}}{r} - \Gamma$; therefore, the ſum of the above expreſſions may be eaſily found $= \frac{S \,.\, \overline{r-1} \,.\, \overline{BC-ABC}}{r} + R + W - M$.

If the three lives be of equal age, the value of the reverſion will be $= \frac{2S \,.\, \overline{r-1}}{3^r} \times \overline{V-L}$. This expreſſion may be derived either from the foregoing general rule, or immediately from

from the different series, and is known to be accurately true from self-evident principles. Q. E. D.

I have now given general rules for determining the values of reversions depending upon three lives in every case which, as far as I can discover, will admit of an *exact* solution. The remaining cases, which are nearly equal in number to those I have investigated, involve a contingency for which it appears very difficult to find such a general expression as shall not render the rules much too complicated and laborious. The contingency to which I refer is that of *one life's failing after another in any given time.* The fractions expressing this probability are every year increasing, so that the value of the reversion must be represented by as many series at least as are equal to the difference between the age of one of the lives, and that of the oldest life in the table of observations. I have indeed so far succeeded in the method of approximation as that the reversion may be generally ascertained within about $\frac{1}{50}$th part of its exact value; but I shall not trouble the Royal Society at present with these investigations.

The 34th, 35th, and 36th problems in Mr. SIMPSON's Select Exercises involve this contingency, and, by the assistance of M. DE MOIVRE's hypothesis, admit of an easy solution. But such is the fallacy of this hypothesis, that it renders Mr. SIMPSON's conclusions obviously wrong, though his reasoning is perfectly correct: for it cannot surely be an *equal* chance *in all cases* that one life shall die after another. In the short term of a single year the chances are indeed so nearly equal, that it would be wrong to perplex the solution by attempting

 greater

greater accuracy. But when the number of years, and the difference between the ages of the two lives are, considerable, those chances must vary in proportion; and, therefore, unless the contingency is blended with another which shall very much diminish the probability of the event, the solution, by thus indiscriminately supposing the chances to be equal, must be rendered extremely inaccurate. In Mr. SIMPSON's 36th problem the solution by this means appears to be absurd: for, in the particular case in which C is the oldest of the three lives, the value of the reversionary annuity becomes $=\frac{C-AC}{2}$; that is, the value of an annuity in this case during the life of C after B and A, provided A dies first, is the same whatever be the age of B; for no mention is made of his life in the foregoing expression. It should be observed, however, that the rule itself is strictly true, and that the error arises from Mr. SIMPSON's having been misled by the hypothesis in determining the probability of B's dying after A in his investigation of the 34th problem, which is applied to the solution of this problem *.

I have declined giving specimens of the different values of the reversions as deduced from the foregoing rules and those which have been hitherto published, not only from an apprehension of becoming tedious, but also from the conviction that at present they are unnecessary; those which I have formerly given being, I think, sufficient to prove the inaccuracy of M. DE MOIVRE's hypothesis. In those instances in which I have compared some of the foregoing rules with the approxi-

* It is proper to observe, that I have followed Mr. SIMPSON's method of determining this contingency in the 23d, 27th, 28th, and 29th Problems in my Treatise on Annuities.

mations now in uſe, I have invariably found the latter to be erroneous; nay, in ſome caſes, the values were almoſt twice as great as they ought to have been. This is particularly true when one of the lives is very young, and both or either of the other lives are very old. In reverſions of this kind I believe that this is generally the caſe, and that it ſeldom happens that the ages of the three lives are nearly equal. The approximations therefore can hardly ever be uſed with ſafety, and it will certainly be moſt prudent not to have recourſe to them when the correct values can be obtained. Should the difficulties attending the ſolution of the remaining problems which involve three lives be ſurmounted (and the taſk may not perhaps be impoſſible), the hypotheſis of an equal decrement of life, as far as it relates to any uſeful purpoſe in the doctrine of annuities, may then be totally abandoned. Or ſhould it even be found impracticable to deduce ſolutions of thoſe problems which are ſtrictly and accurately true; yet, I am ſatisfied from my own experience that ſuch near approximations may be procured as to render this hypotheſis equally unneceſſary.